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ON SHELL SOLUTIONS FOR MASONRY DOMES

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Abstract—Membrane solutions are sensitive to small deformations of the surface of a shell, and, as such, are not suitable for the analysis of the type of dome considered here. Earlier studies [3, 4] on the application of plastic theory to masonry construction are extended to the analysis of shells; some brief comments are made on the structure of St. Sophia.

INTRODUCTION

A SHELL is usually considered to be "thin" when the thickness (t) is less than about 5 per cent of the local (and, of course, varying) radius of curvature (R); certainly a shell having $t/R = 0{\cdot}01$ will be thin, and it is thought that a membrane analysis will give accurate estimates of the shell stress resultants, except in the neighborhood of boundaries, or for certain (unfortunate) shapes of shell. In a membrane analysis it is assumed that all stress resultants act in the middle surface of the shell, and that bending does not occur; thus, for example, the general forces acting on an element of a shell of revolution will be as shown in Fig. 1.

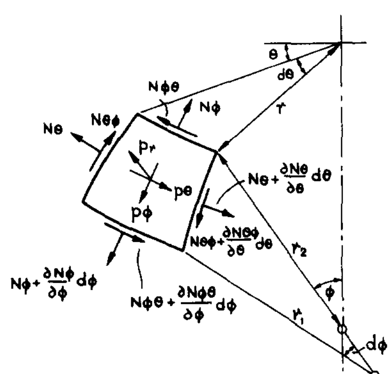

FIG. 1. Shell element (after Flügge).

For such a shell, three equations for the three stress resultants may be written by resolving in the θ and φ directions and radially; these equations will involve the load components p_θ, p_φ and p_r (the presentation here follows Flügge [1]). Thus this particular problem is statically determinate, since the stress resultants can be determined (to within arbitrary functions of integration to be found from the boundary conditions) without reference either to compatibility conditions or to material properties; the three equations of equilibrium suffice to determine the three stress resultants.

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As an example, a shell of revolution subject only to its own weight will have a stress system independent of θ; in Fig. 1, $p_\theta = 0$ and $N_{\theta\varphi} = N_{\varphi\theta} = 0$. The two remaining equations of equilibrium become:

$$\frac{d}{d\varphi}(rN_\varphi) - r_1 N_\theta \cos\varphi + p_\varphi r r_1 = 0$$

and (1)

$$\frac{N_\varphi}{r_1} + \frac{N_\theta}{r_2} = p_r$$

where

$$\begin{aligned} r &= r_2 \sin\varphi \\ p_\varphi &= p \sin\varphi \\ p_r &= -p\cos\varphi \end{aligned} \tag{2}$$

and r_1 and r_2 are the two radii of curvature as defined in Fig. 1; p is the weight per unit area of the shell.

The solution for $r_1 = r_2 = R$ (spherical dome) is particularly simple, and leads to

$$N_\varphi = \frac{-pR}{1+\cos\varphi} \tag{3}$$

the value of N_θ being determinable from the second of equations (1), $N_\varphi + N_\theta = Rp_r = -pR\cos\varphi$. (The expression for N_φ could have been found directly from consideration of the equilibrium of a cap of the dome, Fig. 2.)

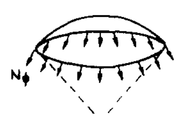

FIG. 2. Cap of dome.

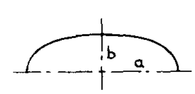

FIG. 3. Ellipsoidal cross-section.

If now the cross-section of the dome is not circular, but elliptical, as shown in Fig. 3, so that the shell is an ellipsoid of revolution, Flügge gives the following expression for the stress resultant N_φ (corresponding to equation (3) for the sphere):

$$N_\varphi = \frac{-pb^2}{2}\frac{(a^2\sin^2\varphi + b^2\cos^2\varphi)^{\frac{1}{2}}}{\sin^2\varphi}\left[\frac{1}{b^2} - \frac{\cos\varphi}{a^2\sin^2\varphi + b^2\cos^2\varphi} + \frac{1}{2ac}\ln\frac{(a+c)(a-c\cos\varphi)}{(a-c)(a+c\cos\varphi)}\right] \tag{4}$$

where $c^2 = a^2 - b^2$.

To compare the spherical and ellipsoidal solutions, the stress at the crown of the dome ($\varphi = 0$) is given from equation (3) as

$$N_\varphi = -\frac{pR}{2} \tag{5}$$

and from equation (4) as

$$N_\varphi = -\frac{pa^2}{2b} \tag{6}$$

Thus the crown stress in the ellipsoidal shell is augmented, compared with the sphere, by a factor involving a/b, the ratio of the major to minor axes of the ellipse. (It should be noted that these solutions are independent of the precise extent of the shells; as mentioned, the boundaries will introduce local perturbations in the expressions).

Numerically, suppose a spherical domical cap is used over an area of diameter 100 ft, the radius of the shell being also 100 ft, so that the rise is 13·4 ft, Fig. 4. By modern standards this is not a particularly shallow shell, as is confirmed by the relatively low value of ambient stress. Taking a unit weight of material as 144 lb/ft^3 (a number which cancels conveniently with the number of sq. in. in a sq. ft), equation (5) for the sphere, using $R = 100$ ft, gives a crown stress of 50 lb/in^2. (This value may be compared with a usual compressive working stress for concrete of say 1250 lb/in^2, or with the crushing strength of a medium sandstone of say 6000 lb/in^2).

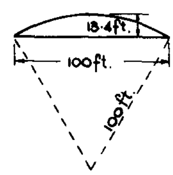

FIG. 4. Spherical dome.

Now suppose that, on striking the shuttering for the shell, small deformations occur as the shell takes up load (or, alternatively, suppose such small deformations to be due to slightly imperfect placing of the original centering), so that the crown sags by 4 in. and the whole dome takes on the form of part of an ellipsoid of revolution (note that 4 in. is $\frac{1}{3}$ per cent of the radius 100 ft). For the dimensions given, this tiny imperfection leads to an ellipsoid having a ratio major to minor axis a/b of 2, with $a = 60$ ft, and the ambient stresses at the crown, equation (6), are increased to 60 lb/in^2.

Put another way, an ellipse, of major to minor axis ratio 2, and a circle can be drawn to coincide to within ± 4 in. over a span of 100 ft and a rise of 13·4 ft. To construct formwork to this accuracy is a formidable task, and the engineer using membrane theory must, if he is aware of the geometrical closeness of an ellipse and a circle, be in some considerable doubt as to the means of his calculations.

When structural calculations appear to be as sensitive to small imperfections as they do in this example, one of two conclusions may usually be drawn. Either the structure itself is of a dangerous type, liable to develop stresses of a magnitude not foreseen by the designer, or the calculations made by the designer are the wrong ones for that particular structure. Now a shell structure of reasonable dimensions (i.e. not too shallow nor too thin) is not particularly dangerous; on the contrary, the survival of the dome of St. Sophia from the sixth century, and of numerous thin Gothic vaults from the twelfth century onwards, most of them so "badly" built as to depart grossly from any recognizable mathematical surface, and subject in addition to accidental deformations due to uneven settlements, indicates that the masonry shell is a particularly stable form of structure. The conclusion must be in this case that conventional membrane calculations have little relevance to the estimation of the behavior of the shell.

This conclusion is reinforced by the numerical calculation made above. In general, the stresses in a smoothly curving thin shell subjected only to its own weight are of the order $R\rho$, where R is the local radius of curvature and ρ the density of the material (the stress is of course, independent of the (constant) thickness of the shell). The type of shell discussed in this paper has R measured in tens of feet; taking ρ again as 144 lb/ft^3, the stresses will therefore be measured in tens of lb/in^2. No problem arises, therefore, of crushing of the material; since *strength* is not here a prime design criterion, the detailed numerical calculation of values of stresses throughout the shell is a meaningless operation.

Equally, *local* instability is no problem. Ahm and Perry [2], quoting other authors, conclude that a reasonable value of local buckling stress σ_{cr} for a dome is given by

$$\sigma_{cr} = 0{\cdot}2E\frac{t}{R} \tag{7}$$

where E is Young's Modulus. Taking E as 3×10^6 lb/in^2, and σ_{cr} as the ambient stress of say 60 lb/in^2, t/R is determined from equation (7) as 10^{-4}. Such thin shells are outside the structural range. (The stresses at Smithfield were of the order 600 rather than 60 lb/in^2, and buckling was significant. It may be noted that a hen's egg has $t/R \sim 10^{-2}$.)

Again, deflexions are not likely to impose limitations on the design, since a reasonably shaped shell is inherently stiff. Thus neither strength, stiffness, nor local stability would seem to govern the design of the type of shell considered here. Two recent papers [3, 4] have explored the application of plastic theory, originally developed for the design of steel framed structures [5, 6], to masonry construction, and it was concluded that in many cases overall stability was the prime criterion for structural design. This work will be developed further here to deal with the problem of masonry domes.

LIMIT ANALYSIS OF MASONRY

It will be assumed that stresses are low enough for the crushing strength of the masonry to be considered effectively infinite. Further, construction will be assumed to be dry, that is, the weak tensile strength of mortar will be ignored. Thus the masonry can sustain any value of compressive stress and zero tensile stress. It will be assumed further that sliding of one stone on another does not occur.

Under these conditions, failure at a cross-section of a masonry member occurs by *hinging* about a free edge of the masonry, and it has been shown [3] that the limit theorems of plastic theory can be applied. The hinging action can perhaps best be illustrated by a simple two-dimensional example of a voussoir arch, Fig. 5. As drawn, this segmental circular arch fits exactly between the abutments. In fact, of course, practical imperfections will lead to an inexact fit, and Fig. 6 illustrates the two cases of the abutments being slightly too wide and slightly too narrow. To accommodate itself to the span, the arch has in either case formed three hinges.

Note that, in conventional terms, the arch has three redundancies, and the formation of three hinges has turned the redundant structure into one which is statically determinate. Thus, in Fig. 6(a), the value of the horizontal thrust H_1 can be found simply by taking moments for the half arch. Such hinging (involving cracking of the masonry construction) is completely harmless from the point of view of stability of the arch; a fourth hinge is necessary before a mechanism of collapse can form, and it may be that the fourth hinge can never occur for the given geometry and loading.

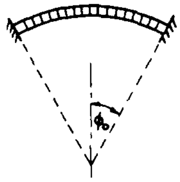

FIG. 5. Voussoir arch.

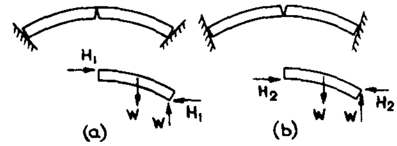

FIG. 6. Cracked voussoir arches.

The thrust line for the arch must of course pass through a hinge point. Thus the thrust lines for the two configurations of Fig. 6 will be somewhat as sketched in Fig. 7. (It remains

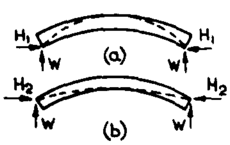

FIG. 7. Limiting thrust lines.

to be shown that thrust lines can in fact be found to lie, as sketched in Fig. 7, within the masonry; this problem is discussed further below.) It is clear from the sketched shapes in Fig. 7 that the values of H_1 and H_2 must be markedly different; numerical calculations for $\varphi_0 = 30°$ in Fig. 5, and for $t/R = 0{\cdot}05$, lead to the values $H_1 = 1{\cdot}27W$ and $H_2 = 2{\cdot}91W$, where W is the weight of the half arch. In this case, therefore, which is analogous to that of the dome of Fig. 4, the stress at the crown of the arch can differ by a factor of more than 2, depending on slight movements of the abutments. The fact that there exist such limiting positions of the thrust line for arches was understood quite early; Coulomb, in his paper of 1773 [7], states clearly that the thrust line can approach the extrados or intrados of an arch, and Breymann [8] gives a detailed discussion of maximum and minimum values of thrust.

The whole problem of the determination of the stability of masonry can, in fact, be reduced to the formally simple question of whether or not a thrust line (thrust surface for the three-dimensional structure) can be found lying wholly within the masonry. With the assumption of zero tension in the masonry, it is inadmissible for the thrust line to lie outside the stonework, but if *any* thrust line within the masonry can be found which is in equilibrium with the externally applied loads, then the "safe" theorem of limit design ensures that the structure will be stable. Thus either Fig. 7 (a) or Fig. 7 (b) could be used as complete proof that the arch, as sketched, will be stable. These thrust lines are not necessarily those that would develop in practice, and the horizontal thrust H might assume any value between the limiting values H_1 and H_2, but there is no possible value of H within those limits that could cause the arch to collapse.

Now it is well-known that a circular arch of the type sketched in Fig. 5 is of the "wrong" shape to carry its own uniformly distributed load; the shape should be that of an inverted catenary. However, many catenaries can be contained within the real thickness of a circular arch, and the two limiting cases are precisely those sketched in Fig. 7. Guided by this observation, the problem of shell analysis, or of the two-dimensional analogue, arch analysis, can be rephrased.

Membrane solutions are certainly required, in the sense that a thrust surface (or the two-dimensional thrust line), is required. The membrane should not, however, be constrained to coincide with the centre surface of the original structure. Instead, the geometry of the structure merely defines a region, possibly quite narrow, within which a membrane solution must be established if the structure is to be stable.

THE VOUSSOIR ARCH

Still working with the two-dimensional arch as an analogue of the shell, the ideas can perhaps be clarified by making some limiting calculations. Figure 8 shows a circular arch of mean radius R embracing an angle $2\varphi_0$; the thickness of the arch has been chosen so that a line of thrust can only just be contained wholly within the masonry. Thus the thrust line touches the crown of the arch at the extrados, and also passes through the extrados at the springings; at some angle β (to be determined) the thrust line touches the intrados.

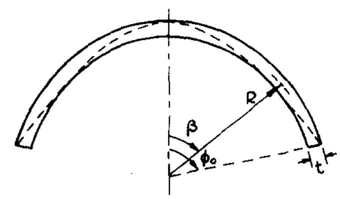

FIG. 8. Voussoir arch of minimum thickness.

The centre line of the arch is sketched in Fig. 9, with the thrusts offset by a distance $t/2$; if the weight per unit length of arch is k, the weights $k\beta$ and $k(\varphi_0 - \beta)$ will act as shown. Three equations of equilibrium may be written for each of the two portions of Fig. 9, and the four unknowns H, X, Y and Z eliminated, leaving two equations connecting t/R, β, and φ_0. Thus for any value of the cut-off angle φ_0, the value of t/R may be determined.

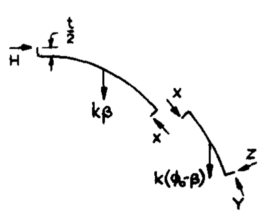

FIG. 9. Elements of arch.

Figure 10 gives the resulting plot of the thickness ratio required for the arch just to be stable. A full semicircular arch requires t/R to be just greater than 10 per cent, but a 5 per cent arch can be used up to a cut-off angle φ_0 of about 75°. (These brief calculations will be used below in the discussion of certain aspects of the behavior of domes).

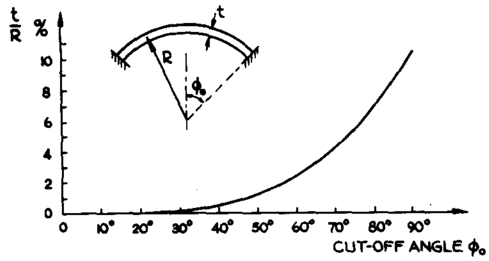

FIG. 10. Required thickness of arch.

A physical interpretation may be given for this limiting position of the thrust line. Where the thrust line touches the surface of the masonry an incipient hinge develops, and, corresponding to the sketch of Fig. 8, hinges will form as shown in Fig. 11. These hinges are sufficient in number to allow a collapse mechanism for the whole arch; indeed, only four hinges (of the proper signs, i.e. opening and closing alternately when viewed from say the intrados) are necessary for collapse, and the fifth is formed due to the symmetry of the structure and the loading.

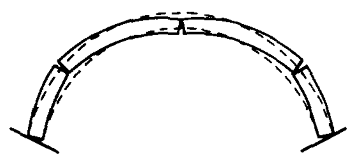

FIG. 11. Collapse mechanism for arch.

Thus, corresponding to the analytical problem of finding a thrust line lying wholly within the masonry, there exists a physical problem of arranging a sequence of hinges to form a collapse mechanism for the structure. The problem of the collapse mechanism is essentially one of geometry, and it may well be that the geometrical configuration of a particular masonry element is such that there is no possible pattern of hinges that will lead to a mechanism of collapse. This is true, for example, for a well designed flying buttress [3], in which case a thrust line can always be found to lie within the masonry, and the absolute statement can be made that only decay of the stonework will lead to collapse of the buttress.

THE DOME

Parsons [9] gives the weight of the dome of Santa Maria del Fiore, spanning more than 140 ft, as 70,000,000 lb; this corresponds to about 2000 lb/ft^2 of the dome surface. The much thinner (about 2 ft 6 in.) main dome of Saint Sophia has a unit weight of an order of magnitude less. These figures are to be compared with a unit wind pressure of typical magnitude 20 lb/ft^2. Wind forces are therefore small or very small compared with the self-weights of the structures considered here, and wind will be ignored in all of the following calculations.

The spherical membrane solution, equation (3), will first be examined; under self-weight alone, the two stress components are

$$
\begin{aligned}
N_\varphi &= -\frac{pa}{1+\cos\varphi} \\
N_\theta &= -pa\left(\cos\varphi - \frac{1}{1+\cos\varphi}\right)
\end{aligned}
\tag{8}
$$

Considering first a full hemisphere, the value of the stress resultant N_φ, which is always compressive, increases from $\frac{1}{2}pa$ at the crown ($\varphi = 0$) to pa at the springing ($\varphi = 90°$). In contrast, the value of N_θ, which is compressive at the crown (of value $\frac{1}{2}pa$) changes sign at $\varphi = 51{\cdot}8°$ (the solution of $\cos^2\varphi + \cos\varphi = 1$, i.e. $\cos\varphi = \frac{1}{2}(\sqrt{5}-1)$). For $\varphi > 51{\cdot}8°$, tensile stresses are developed in the hoop direction, as sketched for the half dome in Fig. 12.

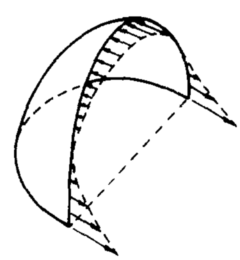

FIG. 12. Half dome.

Such tensile stresses are inadmissible for the material considered here, and, indeed, for the general masonry structure; the presence of iron hoops and chains towards the springing of existing domes testify to the concern that engineers have felt on this score. (It is of interest that Poleni [10], in his discussion of the stability of the dome of St. Peter's, Rome, advocated the use of such tie rings, although he had come to the conclusion that despite cracking, the dome would continue to stand without ties.) Thus the membrane solution, given by equations (8), is inadmissible for the full hemispherical unreinforced dome.

There is no difficulty, of course, for a spherical cap whose cut-off angle φ_0 is less than $51{\cdot}8°$. Providing such an incomplete dome is supported at its base in the proper way, the stresses will be purely compressive. The thickness of such a dome is then governed theoretically only by the need to prevent local instability, and any practical dome will have a thickness dictated by constructional requirements. For the hemisphere, however, and in general for the dome of cut-off angle φ_0 greater than $51{\cdot}8°$, a certain thickness is required to contain an all-compressive thrust surface, departing slightly from the hemisphere, but still lying within the masonry.

Following Poleni, the dome may be cut into a series of "orange slices", Fig. 13, and the stability of each of these segments investigated separately; if it can be shown that each element of the sliced structure is stable, then the safe theorem of limit design can again be used to demonstrate that the original structure must be stable. The analytical problem is very similar to that of the voussoir arch; now, however, the width of the arch tapers to zero at the crown, so that the weight is not uniformly disturbed. A thrust line is

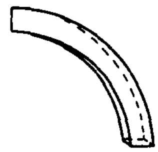

FIG. 13. "Orange slice" of dome.

to be sought for the segment, lying within the masonry; as before, the limiting case will be studied when a collapse mechanism is just formed, giving the minimum thickness of dome.

In formal mathematical terms, the hoop stress N_θ of equations (1) is set equal to zero, and r_1 is left for the moment undefined; equations (1) then solve to give

$$prr_1 \cos^2\varphi = \text{const.} \tag{9}$$

as the intrinsic equation for the thrust surface. The form of equation (9) is not particularly convenient for the purpose of comparing with the spherical surface, and the techniques used above for the voussoir arch and illustrated in Figs. 8 and 9 will be applied to the segment of Fig. 13.

Although each segment is considered in isolation, the collapse mechanism corresponding to the limiting position of the thrust line must be a possible mechanism for the dome considered as a whole. Further, it must be demonstrated that the thrust line, while just reaching the surface of the masonry at a number of discrete points, the hinges, elsewhere lies always within the masonry. The correct limiting position of the thrust line for the orange slice is sketched in Fig. 14; it touches the extrados of the dome at P and the intrados at Q. Figure 15 shows the corresponding collapse mechanism, for which a central region near the crown of the dome does not deform, but merely descends vertically; adjacent segments will, of course, move apart between P and the base as the dome collapses, expressing the condition of zero tension in the masonry.

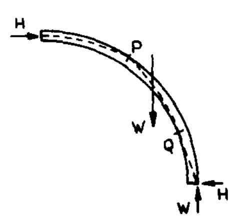

FIG. 14. Orange slice of minimum thickness.

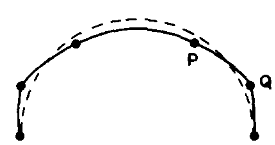

FIG. 15. Collapse mechanism for dome.

Note in Fig. 14 that the thrust H acting on the segment at the crown must be provided by an equal and opposite thrust on the diametrically opposite segment; the analysis applies, therefore, only to a complete symmetrical dome, and not, for example, to the half dome of Fig. 12.

In Fig. 14, the sections from the crown to P, from P to Q, and from Q to the springing, may be considered exactly as in Fig. 9 for the arch; alternatively, a virtual work analysis may be made. In either case, equations may be solved numerically for the required thickness t/R in terms of the cut-off angle φ_0 for the dome; Fig. 16 gives a plot of this thickness. For the full hemisphere, $\varphi_0 = 90°$, the required thickness is 4·20 per cent of the radius;

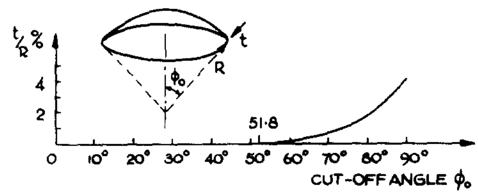

FIG. 16. Required thickness of dome.

as φ_0 is reduced, the required thickness drops, and becomes theoretically zero at $\varphi_0 = 51{\cdot}8°$. (The angular co-ordinates, say β and β', of points P and Q in Fig. 14 are always related by the expression

$$(1-\cos\beta)\tan\beta' = (1-\cos\beta')\tan\beta \tag{10}$$

The maximum of the expression $(1-\cos\beta)/\tan\beta$ is given by

$$\cos\beta = \tfrac{1}{2}(\sqrt{5}-1), \qquad \text{i.e. } \beta = 51{\cdot}8°).$$

Thus a full hemispherical dome, whose thickness is say 5 per cent of the radius, and constructed of masonry incapable of supporting tension, will nevertheless stand without reinforcement. Poleni's conclusion about St. Peter's is justified (Siegel [11] gives the span of St. Peter's dome as 40 m, and the shell thickness as 300 cm, a t/R ratio of 15 per cent; the dome has, of course, two skins from the crown round to $\varphi \sim 70°$, the individual t/R ratios for the skins being about 6 and 8 per cent.) The unreinforced dome must, however, be supported properly at the base; Fig. 14 shows the inward thrust H that must be applied as well as the vertical force W.

A good estimate of the value of H for the hemisphere may be made by considering the overall equilibrium of the segment, redrawn in Fig. 17. The line of action of the weight W of the segment is as shown, from which $H = (1-\pi/4)W = 0{\cdot}215W$. Thus for a hemispherical masonry dome spanning 100 ft and weighing say 5,000,000 lb, there will arise a total horizontal force H, uniformly distributed round the base, of value about 1,000,000 lb, i.e. some 3,000 lb per lineal foot. This force might be provided by a tie ring at the base; in the absence of such a tie, proper abutments must be provided in the construction to absorb this thrust. Two types of such abutments are found in St. Sophia, of which a brief discussion is given below.

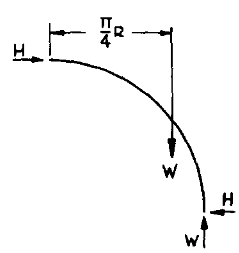

FIG. 17. Equilibrium of orange slice.

Before making further analyses, it is of interest to review in the light of conventional limit design theory the work just completed on the dome. A thrust surface has been found,

Fig. 14, in equilibrium with the applied loads (self-weight of the dome) and lying wholly within the masonry; corresponding to this position of the thrust surface there exists a mechanism of collapse, Fig. 15, and the calculations lead to a definite thickness of dome for these conditions to hold. According to the basic theorems of limit design, the value of the derived thickness is unique; further, a dome of greater thickness cannot collapse under its own weight, and an unreinforced dome of lesser thickness cannot be built.

INCOMPLETE DOMES

The crown of a dome can, of course, be completely omitted (i.e. the dome can have an "eye") without any apparent structural alteration being necessary. Similarly, a heavy lantern (some 500 ton at Florence) may be loaded on to the crown. In either case, a simple analysis can again be made by cutting the dome into orange slices, and constructing thrust lines similar to that of Fig. 14.

Of more interest is the half dome, Fig. 12. The membrane equations admit only of the symmetrical solution of equations (8) for the full hemisphere, and are valid for the half dome only if the vertical edge is subjected to the compressive and tensile forces shown. Thus no spherical membrane solution exists for the free-standing half dome. Nevertheless it is intuitively clear that a half dome of sufficient thickness *will* stand, and a safe estimate of this thickness may be made by again using the technique of slicing the dome into sections.

Suppose, for example, that the half dome is sliced into a series of parallel semicircular arches, Fig. 18, each arch being independent of its neighbours. From Fig. 10 for the voussoir

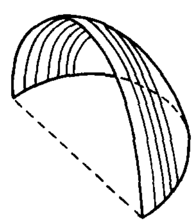

FIG. 18. Sliced half dome.

arch, if each ring has a thickness t/R of just over 10 per cent, then it will be stable. Similarly, if the half dome has a cut-off angle of 75°, so that the sliced arches do not form full semicircles, then a 5 per cent dome will be stable. These thickness estimates are safe, since it has been assumed that each arch ring acts independently; the interconnection in the real structure ensures, by the safe theorem of limit design, that a thinner shell will actually stand.

Indeed, the parallel ring slicing of Fig. 18 must destroy quite large interconnecting forces between the rings, and hence lead to a stress solution which is quite far from the actual stresses in the real dome; nevertheless, the parallel rings lead to a stress system which is in equilibrium with the self-weight of the masonry, and hence which is admissible for this form of analysis. In practice, the lack of support along the vertical free edge (c.f. Fig. 12 showing the membrane forces) is bound to cause the dome to sag forward slightly if there is any imperfect fit between the voussoirs. Mainstone [12] discusses such sagging in the half domes of St. Sophia (see below) and attributes the movement to the very slow-drying beds of thick mortar between the bricks.

Thus a half dome, of sufficient thickness, will stand as a free structural element. It is clear from Fig. 12, however that a half dome would form a good buttress, since the free vertical edge can be subjected, at least round the top half, to large compressive forces. As an apparently trivial example, consider a complete dome, shown in plan in Fig. 19, to be first cut into two equal free-standing half domes, and then one of the half domes further cut into orange slices. Each orange slice will be stable if, as mentioned above and shown

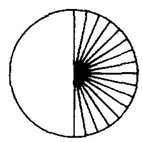

FIG. 19. Sliced full dome.

in Fig. 14, a propping force H is applied at the crown. Now these propping forces H summed for all the slices will have an out-of-balance component acting on the other half dome. By definition, the whole dome stands, and hence the half dome subjected to the out-of-balance force (of magnitude 0·068 W, where W is the weight of the whole dome) will also stand.

A half dome can therefore (if of sufficient thickness) either stand freely, or stand when "leaned against" by an equal half dome. Now it can be proved, and is intuitively "obvious", that if a structure of the type considered here is subject to a set of static forces and a single variable force, say P, and if the structure is stable under a value P_1 of the variable force and also stable under a value P_2 of the variable force, then it will be stable under any value P of the variable force lying between the limits P_1 and P_2. Thus, since the half dome will stand against zero lateral thrust at the crown (free standing) and against the full lateral thrust of another half dome, it will also stand against any intermediate thrust.

As a structural curiosity, the three-quarter dome shown in plan in Fig. 20 will be considered. In this figure, the dome has been divided into a half dome and two "eighth" domes, the eighth domes being in turn divided into orange slices. It will be seen that,

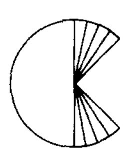

FIG. 20. Sliced three quarter dome.

comparing Fig. 20 with Fig. 19, one quarter of the complete dome has been removed; the remaining orange slices in Fig. 20 will, of course, thrust against the half dome with a force less than that exerted by another half dome. Thus, if the thickness of the masonry is such that a half dome would stand freely, then a three-quarter dome will also stand.

The difficulties in the exact analysis of incomplete domes lie in their asymmetry. A safe analysis requires the construction of a three-dimensional thrust surface lying within the masonry, and satisfying the loading boundary conditions; the geometry of such a surface can be complex. It is for this reason that two-dimensional slices can lead to much simpler solutions; although such solutions are safe, they may be so far from reality as to give too conservative results. An alternative approach would be to write the virtual work equation for assumed possible collapse mechanisms; besides leading to unsafe solutions, this technique continues to have the disadvantage that the geometry of deformation is difficult.

For the actual structures considered here, having t/R ratios of 5 per cent or more, the safe technique of slicing is perfectly adequate; but the technique might be too "blunt" to deal with other, thinner, shells.

ST. SOPHIA

As has been seen, a dome requires buttressing round its periphery, and several different methods are possible. Choisy [13, 14] discusses the Byzantine solutions, of which the three main combinations are shown in Fig. 21. In each of these figures, a circular dome is erected

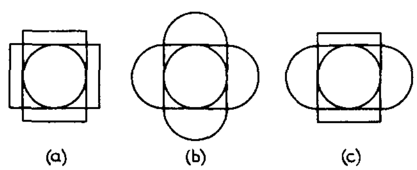

FIG. 21. Byzantine buttressing (after Choisy).

on a square plan, and the four faces of the square are buttressed by arches in (a), by half domes in (b), and by arches on two faces and domes on two faces in (c). St. Sophia has this mixed type of buttressing; Choisy's horizontal section [13] at a level above the side aisles, reproduced in Fig. 22, shows the two arches A and the two half domes C, these half domes being themselves buttressed by much smaller half domes (exedrae). It will be seen that massive (hollow) buttresses support the arches A; Fig. 23 shows Choisy's isometric drawing [14]. This combination of arch and half dome buttresses gives a longitudinal emphasis to the church, the completely open space (flanked by aisles) being 220 by 107 ft.

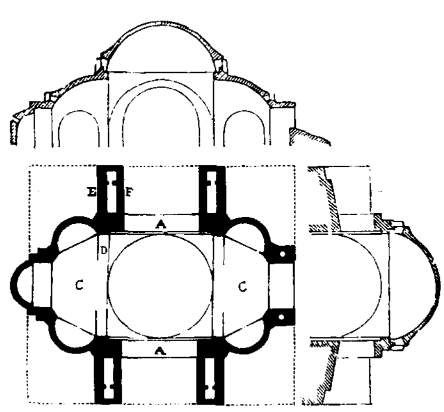

FIG. 22. Choisy's plan of St. Sophia.

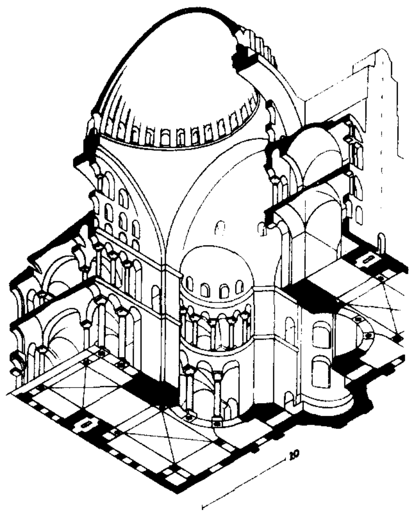

FIG. 23. Choisy's isometric of St. Sophia.

The span of the central dome is approximately 30 m, and the thickness 75 cm, so that the nominal ratio t/R is about 5 per cent. The buttressing half domes have the same dimensions, and the exedrae domes are about 37·5 cm thick. (These figures are from Mainstone [12], based on the work of van Nice, of which some advance information has been published [15, 16].) In fact the domes are far from being semi-circular in section, and are indeed fairly flat; in addition, considerable thickening can be seen in Figs. 22 and 23 towards the peripheries, which are in turn supported by rings of small individual external buttresses.

Thus the cut-off angle φ_0 of the main dome and of the buttressing half domes of St. Sophia is relatively small, and a nominal value of t/R of 5 per cent is completely adequate to ensure the stability of the structure under normal conditions. Some doubt might be felt about the capacity of the buttressing arches, which must carry not only part of the vertical weight of the main dome, but also the lateral thrust of the dome. In fact, St. Sophia could have been built with even narrower arches, being buttressed only by the half domes, Fig. 24. Here the main dome has been sliced into parallel rings, exactly as in the study of the half dome, Fig. 18, and it will be seen that an equilibrium solution can be constructed involving lateral thrusts against the buttressing half domes only.

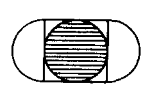

FIG. 24. St. Sophia with east and west buttresses only.

Such a modified structure would have had lowered resistance to earthquake conditions, and the severe earthquakes of 986 and 1346 in fact demonstrated the capacity of the actual buttressing arches to resist lateral thrust. The first of these earthquakes led to the collapse of the western buttressing half dome together with one quarter of the main dome, the remaining three quarters of the main dome standing. In this condition, the north and south arches would have been subjected to considerable lateral thrust by the three quarter dome. The structure was repaired, and the second severe earthquake caused collapse of the eastern buttressing half dome together with the corresponding quarter of the main dome.

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Résumé—Les solutions 'membrane' sont sensibles aux petites déformations de la forme d'un coque, et ne peuvent donc pas être utilisées pour l'analyse du genre de coupole considerée ici. Les moyens d'appliquer les méthodes plastiques à la construction en maçonnerie [3, 4] sont élargies pour servir à l'analyse des coques; certaines remarques concernant la structure de St. Sophie sont inclues.

Zusammenfassung—Lösungen für Membranschalen sind sehr empfindlich gegen Verformungen der Hüllenoberfläche, daher ungeeignet zur Untersuchung von Kuppeln. Frühere Untersuchungen [3, 4] der Anwendung plastischer Theorien bei Mauerwerkskonstruktion werden erweitert und zur Untersuchung von Hüllen angewendet. Einige Bemerkungen über die Konstruktion von St. Sophia werden gemacht.

Абстракт—Решения мембраны чувствительны к маленьким деформациям поверхности оболочки, и, как таковые не годятся для анализа типа купола, обсуждаемого здесь. Ранние изучения [3, 4] применения пластической теории к каменной конструкции, распространяются на анализ оболочек.

Сделано несколько кратких замечаний о структуре Св. Софии.